Faster Mixing of the Jerrum-Sinclair Chain

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Based on joint work with:

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- Zhe Ju @ NJU
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- Yitong Yin @ NJU
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Backgrouds and results

Approximate counting/sampling

Given a function $\mathrm{wt} : \{0,1\}^n \to \mathbb{R}_{\geq 0}$:

exact counting

compute the partition function

$$Z := \sum_{x \in \{0,1\}^n} \mathrm{wt}(x)$$

usually very hard

approximate counting

Find an estimation $\widehat{Z}$ such that

$$(1-\varepsilon)Z \leq \widehat{Z} \leq (1+\varepsilon)Z$$

$\Updownarrow$ Jerrum-Valiant-Vazirani'86

exact sampling

draw sample X from distribution

$$\mu := \frac{\mathrm{wt}}{Z}$$

approximate sampling

draw a random $X \in \{0,1\}^n$ such that

$$\|\mu - \mathrm{Law}(X)\|_{\mathrm{TV}} \leq \varepsilon$$

Examples including: (perfect) matchings, independent sets, spanning trees

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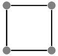
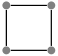
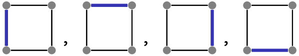
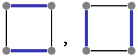
Matchings (monomer-dimer model)

Let $G = (V, E)$ be a simple graph and $\lambda > 0$ be the edge weight

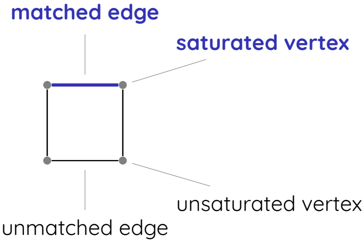
- weight function: $\forall M \subseteq E$, $\mathrm{wt}(M) := \lambda^{|M|}\mathbf{1}[M \text{ is a matching}]$
- partition function: $Z := \sum_M \mathrm{wt}(M)$
- Gibbs distribution: $\mu := \mathrm{wt}/Z$

Matchings of  :

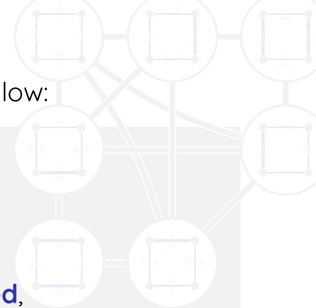
-  1
-  λ
-  λ^2

$$Z = 1 + 4\lambda + 2\lambda^2$$

Some terminologies



Jerrum-Sinclar chain



The Jerrum-Sinclair chain P_{JS} updates a matching X_t to X_{t+1} as follow:

1. select $e = \{u, v\} \in E$, u.a.r.
2. propose a new matching M from following exclusive cases:
   - ($\uparrow$) if u and v are **unsaturated**, let $M \leftarrow X_t \cup \{e\}$
   - ($\downarrow$) if $e \in X_t$, let $M \leftarrow X_t \setminus \{e\}$
   - ($\leftrightarrow$) if one end point is **unsaturated** and the other is **saturated**, say u is saturated by edge e' and v is **not**, let $M \leftarrow X_t \setminus \{e'\} \cup \{e\}$
   - ($\perp$) otherwise, let $M \leftarrow X_t$
3. with probability $\min\{1, \mu(M)/\mu(X_t)\}$ set $X_{t+1} \leftarrow M$; otherwise, set $X_{t+1} \leftarrow X_t$

Lazy Jerrum-Sinclair chain$^{z^{z^z}}$

$$P_{zz} := \frac{1}{2}(P_{\mathrm{JS}} + I)$$

MCMC method: run P_{zz} for t steps (sufficiently large); then output X_t

Mixing time: $T_{\mathrm{mix}}(P_{zz}) := \max_x \min\left\{ t \;\middle|\; \left\| P^t_{zz}(x, \cdot) - \mu \right\|_{\mathrm{TV}} \leq \frac{1}{100} \right\}$

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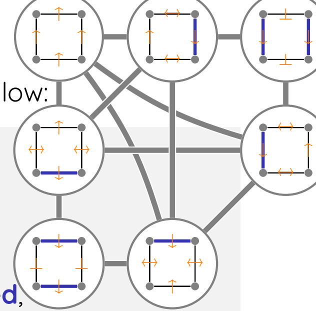

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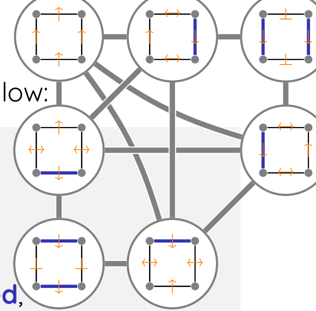

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Results for mixing

$(n = |V|, m = |E|)$

Jerrum-Sinclair'89: General graph; constant $\lambda > 0$,

$$T_{\text{mix}}(P_{zz}) = \widetilde{O}(n^2 \cdot m)$$

(via canonical path)

Chen-Liu-Vigoda'21: Graph with max degree Δ; constant $\lambda > 0$

$$T_{\text{mix}}(\text{Glauber dynamics}) = O(\Delta^{\Delta^2} m \log n)$$

(strong spatial mixing + spectral independence)

Glauber dynamics $\approx P_{zz}$ only allows $(\downarrow)$ and $(\uparrow)$ transitions

C-Yang-Yin-Zhang'24: Graph max degree Δ and girth $= \Omega(\sqrt{\Delta} \log \Delta)$; constant $\lambda > 0$

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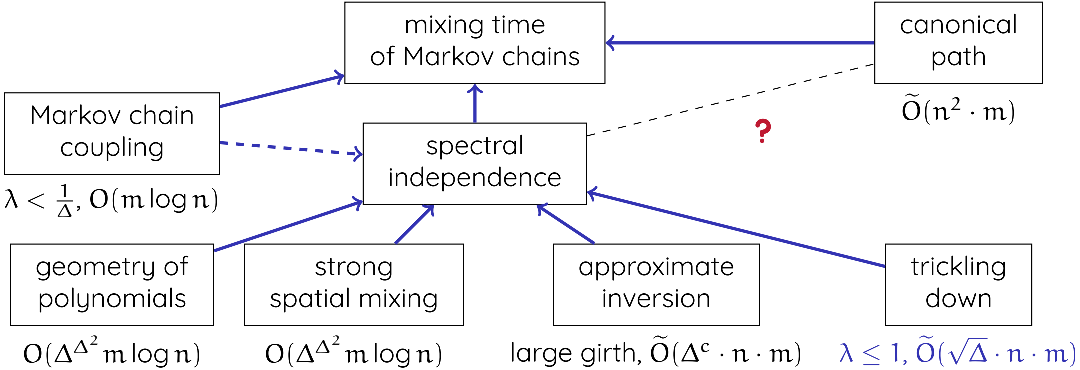
Results for mixing

Techniques with best kown mixing time bounds applying them alone or with SI



Our result

$$T_{mix}(P_{zz}) = O(\Delta m \cdot \min\{n, \Delta \log \Delta \cdot \log n\}) = \widetilde{O}(\Delta^2 \cdot m)$$

Corollary:

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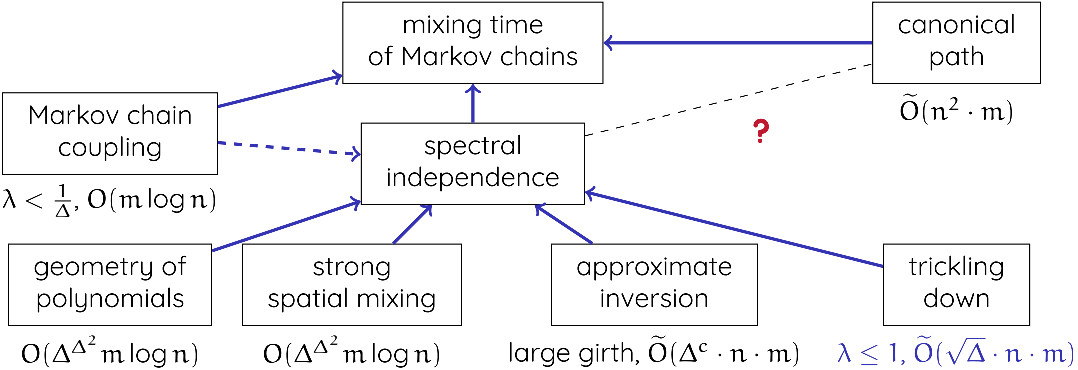

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Mixing time analysis via local functional inequalities

Functional inequalities

Distribution μ over $\Omega \subseteq 2^E$

Random variable $F = f(X)$ for $f : \Omega \to \mathbb{R}_{\geq 0}$ and $X \sim \mu$

entropy

$$\mathrm{Ent}[F] := \mathbb{E}[F(\log F - \log \mathbb{E}[F])]$$

variance

$$\mathrm{Var}[F] := \mathbb{E}[F^2] - \mathbb{E}[F]^2$$

Inner product for functions $f, g : \Omega \to \mathbb{R}_{\geq 0}$: $\langle f, g \rangle_\mu := \mathbb{E}[f(X)g(X)]$

Dirichlet form Markov chain P with stationary distribution μ

$$\mathcal{E}_P(f,f) := \langle f, (I-P)f \rangle_\mu \underbrace{=}_{\text{for reversible chains}} \frac{1}{2} \sum_{x,y \in \Omega} \mu(x)P(x,y)(f(x)-f(y))^2$$

log-Sobolev inequality

$$\forall f \in \mathbb{R}^{\Omega}_{\geq 0}, \quad \rho(P) \cdot \mathrm{Ent}[F^2] \leq \mathcal{E}_P(f,f)$$

$$T_{\mathrm{mix}}(P_{zz}) = \rho(P_{JS})^{-1} \times \underbrace{O(\log n)}_{O(\Delta^2 m)}$$

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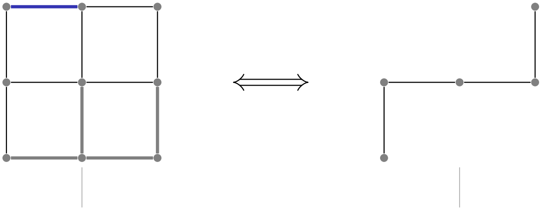
Pinnings

Pinning: τ is a 0-1 vector in $\{0,1\}^\Lambda$ ($\Lambda \subseteq E$) which indicates an event that

$$\forall e \in \Lambda, \quad \begin{cases} e \text{ is } \textbf{matched}, & \tau_e = 1 \\ e \text{ is } \textbf{unmatched}, & \tau_e = 0 \end{cases}$$

Feasible pinning: if $M(\tau) := \{e \in \Lambda \mid \tau_e = 1\}$ is a matching

Conditional distributions induced by pinning: $\mu^\tau := \mu(\cdot \mid \tau)$



μ^τ

monomer-dimer model
on the new graph

In particular, $\mu^\varnothing = \mu$

Local functional inequalities

Family of chains:

$$\mathfrak{Q} := \left\{ Q^\tau \;\middle|\; \begin{array}{c} \tau \text{ is a feasible pinning} \\ Q^\tau \text{ has stationary distribution } \mu^\tau \end{array} \right\}$$

Concave Dirichlet forms ✱: $\forall \Lambda \subseteq E, \forall \tau \in \{0,1\}^{E \setminus \Lambda}$, we have

$$\underbrace{\frac{1}{|\Lambda|} \sum_{e \in \Lambda} \mathop{\mathbb{E}}_{c \sim \mu_e^\tau} \left[\mathcal{E}_{Q^{\tau \uplus \{e \leftarrow c\}}}(f,f) \right]}_{\text{average of Dirichlet forms}} \leq \mathcal{E}_{Q^\tau}(f,f)$$

Pinning $\tau \uplus \{e \leftarrow c\}$ extends τ by giving a random edge e a random state c

Local functional inequalities: $\forall \Lambda \subseteq E, \forall \tau \in \{0,1\}^{E \setminus \Lambda}$, let $X \sim \mu^\tau$ and $F = f(X)$,

α-local log-Sobolev inequalities

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local-to-global: + ✱ $\Longrightarrow \rho(Q) \geq \frac{\alpha}{m}$

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Local functional inequalities

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$$\mathfrak{Q} := \left\{ Q^\tau \,\middle|\, \begin{array}{c} \tau \text{ is a feasible pinning} \\ Q^\tau \text{ has stationary distribution } \mu^\tau \end{array} \right\}$$

Concave Dirichlet forms $\boxed{*}$: $\forall \Lambda \subseteq E, \forall \tau \in \{0,1\}^{E\setminus\Lambda}$, we have

$$\underbrace{\frac{1}{|\Lambda|} \sum_{e\in\Lambda} \mathbb{E}_{c\sim\mu_e^\tau} \left[\mathcal{E}_{Q^{\tau\uplus\{e\leftarrow c\}}}(f,f) \right]}_{\text{average of Dirichlet forms}} \leq \mathcal{E}_{Q^\tau}(f,f)$$

Pinning $\tau \uplus \{e \leftarrow c\}$ extends τ by giving a random edge e a random state c

Local functional inequalities: $\forall \Lambda \subseteq E, \forall \tau \in \{0,1\}^{E\setminus\Lambda}$, let $X \sim \mu^\tau$ and $F = f(X)$,

α-local log-Sobolev inequalities

$$\forall f, \quad \frac{\alpha}{|\Lambda|} \sum_{e\in\Lambda} \mathrm{Ent}\left[\mathbb{E}\left[F^2 \mid X_e\right]\right] \leq \mathcal{E}_{Q^\tau}(f,f)$$

local-to-global: + $\boxed{*}$ $\Longrightarrow \rho(Q) \geq \frac{\alpha}{m}$

α-local Poincaré inequalities

$$\forall f, \quad \frac{\alpha}{|\Lambda|} \sum_{e\in\Lambda} \mathrm{Var}\left[\mathbb{E}\left[F \mid X_e\right]\right] \leq \mathcal{E}_{Q^\tau}(f,f)$$

local-to-global: + $\boxed{*}$ $\Longrightarrow \gamma(Q) \geq \frac{\alpha}{m}$

Local-to-global

everything works if **Ent**$[\cdot] \to$ **Var**$[\cdot]$ and $F^2 \to F$

Let $(X_t)_{t=0}^m$ be a Markov chain s.t.

1. $X_0 = \varnothing$
2. given $X_t \in \{0,1\}^{E \setminus \Lambda}$, get X_{t+1} by
   - 2.1 draw $e \in \Lambda$ u.a.r
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Observation: $X_m \sim \mu$

Goal: fix $f : \Omega \to \mathbb{R}_{\geq 0}$, let $F = f(X_m)$, show

$$\mathrm{Ent}[F^2] \leq \rho(Q)^{-1} \mathcal{E}_Q(f,f)$$

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2 Telescoping sum:

$$\mathrm{Ent}[F^2] = \mathrm{Ent}(0) - \mathrm{Ent}(m)$$

$$= \sum_{t=0}^{m-1} \underbrace{(\mathrm{Ent}(t) - \mathrm{Ent}(t+1))}_{=:\triangle_t}$$

3 Law of total ent.:

$$\triangle_t = \mathbb{E}\left[\mathrm{Ent}[F^2 \mid X_t]\right] - \mathbb{E}\left[\mathrm{Ent}[F^2 \mid X_{t+1}]\right]$$

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$$\triangle_t \leq \alpha^{-1}\, \mathbb{E}\left[\mathcal{E}_{Q^{X_t}}(f,f)\right]$$

$$\leq \alpha^{-1} \mathcal{E}_Q(f,f) \quad \text{(concave Diri. form)}$$

2 + 4 : $\mathrm{Ent}[F^2] \leq \alpha^{-1} m \cdot \mathcal{E}_Q(f,f)$

$\implies \rho(Q) \geq \alpha/m$ $\square$

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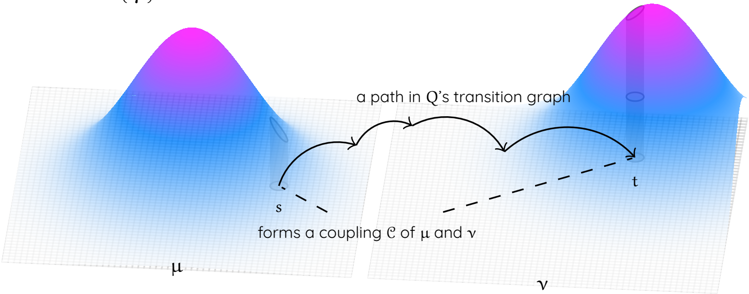
Intuition and establishment of local functional inequalities

Transport flows

Distributions: μ and ν over state space Ω
Markov chain: Q over state space Ω

Transport flow Γ from μ to ν through Q is a distribution over all paths of the transition graph of Q, such that for $\gamma \sim \Gamma$:

- its starting point $s = s(\gamma)$ follows the distribution μ
- its end point $t = t(\gamma)$ follows the distribution ν



Local functional inequalities via transport flow

Theorem

If there is a family of transport flows

$$\left\{\Gamma_e \text{ from } \mu^{e\leftarrow 0} \text{ to } \mu^{e\leftarrow 1} \mid e \in E\right\},$$

s.t. the κ-(strong) expected congestion bound is satisfied: $\forall$ transition $(x \mapsto y)$ of Q,

$$\sum_{e\in E} \mu_e(0)\mu_e(1) \mathop{\mathbb{E}}_{\gamma\sim\Gamma_e} [\ell(\gamma) \cdot \mathbf{1}[(x \mapsto y) \in \gamma]] \leq \kappa \cdot \mu(x)Q(x,y),$$

then the α-local log-Sobolev inequality is satisfied

$$\forall f : \Omega \to \mathbb{R}_{\geq 0}, \quad \frac{\alpha}{m} \sum_{e\in E} \mathrm{Ent}\left[\mathbb{E}\left[F^2 \mid X_e\right]\right] \leq \mathcal{E}_Q(f,f), \quad \text{with} \quad \alpha = \Omega\left(\frac{m}{\kappa \log \frac{1}{\phi}}\right),$$

where $\phi \leq \min\{\mu_e(c) \mid e \in E, c \in \{0,1\}\}$ is the marginal lower bound.

Remark: it is safe to think $\phi \approx \frac{1}{\Delta}$ in our application

Proof outline: local functional inequalities via transport flow

1. Note that $\mathbb{E}\left[F^2 \mid X_e\right]$ is a function of X_e
2. By log-Sobolev ineq. of μ_e by Diaconis and Saloff-Coste'96

$$\mathrm{Ent}\left[\mathbb{E}\left[F^2 \mid X_e\right]\right] \le O\left(\log \frac{1}{\phi}\right) \mathrm{Var}\left[\sqrt{\mathbb{E}\left[F^2 \mid X_e\right]}\right]$$

3. Note that $\Omega(\mu_e) = \{0, 1\}$

$$\mathrm{Var}\left[\sqrt{\mathbb{E}\left[F^2 \mid X_e\right]}\right] = \mu_e(1)\mu_e(0)\left(\sqrt{\mathbb{E}\left[F^2 \mid X_e = 0\right]} - \sqrt{\mathbb{E}\left[F^2 \mid X_e = 1\right]}\right)^2$$

$$= \mu_e(1)\mu_e(0)\left(\sqrt{\mathop{\mathbb{E}}_{\gamma\sim\Gamma_e}\left[f^2(s(\gamma))\right]} - \sqrt{\mathop{\mathbb{E}}_{\gamma\sim\Gamma_e}\left[f^2(t(\gamma))\right]}\right)^2$$

4. Note that $(x, y) \mapsto (\sqrt{x} - \sqrt{y})^2$ is convex on $\mathbb{R}^2$, by Jensen's ineq. on $\mathbb{R}^2$

$$\mathrm{Var}\left[\sqrt{\mathbb{E}\left[F^2 \mid X_e\right]}\right] \le \mu_e(1)\mu_e(0) \mathop{\mathbb{E}}_{\gamma\sim\Gamma_e}\left[f(s(\gamma)) - f(t(\gamma))\right]^2$$

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1 - 4 together:

$$\Omega\left(\frac{1}{\log\frac{1}{\phi}}\right)\mathrm{Ent}\left[\mathbb{E}\left[F^2 \mid X_e\right]\right] \le \mu_e(1)\mu_e(0)\mathop{\mathbb{E}}_{\gamma\sim\Gamma_e}\left[\sum_{(x\mapsto y)\in\gamma}(f(x)-f(y))\right]^2$$

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6 Take summation over e and compare with the Dirichlet form term by term

$$\mathcal{E}_Q(f,f) = \frac{1}{2}\sum_{(x\mapsto y)}\mu(x)Q(x,y)(f(x)-f(y))^2$$

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Proof outline: local functional inequalities via transport flow

1 - 4 together:

$$\Omega\left(\frac{1}{\log\frac{1}{\phi}}\right)\mathrm{Ent}\left[\mathbb{E}\left[F^2 \mid X_e\right]\right] \le \mu_e(1)\mu_e(0) \mathop{\mathbb{E}}_{\gamma\sim\Gamma_e}\left[\sum_{(x\mapsto y)\in\gamma}(f(x)-f(y))\right]^2$$

5 By Cauchy-Schwarz inequality

$$\begin{aligned}\Omega\left(\frac{1}{\log\frac{1}{\phi}}\right)\mathrm{Ent}\left[\mathbb{E}\left[F^2 \mid X_e\right]\right] &\le \mu_e(1)\mu_e(0) \mathop{\mathbb{E}}_{\gamma\sim\Gamma_e}\left[\ell(\gamma)\cdot\sum_{(x\mapsto y)\in\gamma}(f(x)-f(y))^2\right]\\ &= \mu_e(1)\mu_e(0)\sum_{(x\mapsto y)}\mathop{\mathbb{E}}_{\gamma\sim\Gamma_e}\left[\ell(\gamma)\cdot\mathbf{1}[(x\mapsto y)\in\gamma]\right](f(x)-f(y))^2\end{aligned}$$

6 Take summation over e and compare with the Dirichlet form term by term

$$\mathcal{E}_Q(f,f) = \frac{1}{2}\sum_{(x\mapsto y)}\mu(x)Q(x,y)(f(x)-f(y))^2$$

Local functional inequalities via transport flow

If there is a family of transport flows

$$\left\{\Gamma_e \text{ from } \mu^{e\leftarrow 0} \text{ to } \mu^{e\leftarrow 1} \mid e \in E\right\},$$

s.t. the κ-(strong) expected congestion bound is satisfied: $\forall$ transition $(x \mapsto y)$ of Q,

$$\sum_{e \in E} \mu_e(0)\mu_e(1) \mathop{\mathbb{E}}_{\gamma \sim \Gamma_e} [\ell(\gamma) \cdot \mathbf{1}[(x \mapsto y) \in \gamma]] \le \kappa \cdot \mu(x)Q(x,y),$$

then the α-local log-Sobolev inequality is satisfied

$$\forall f : \Omega \to \mathbb{R}_{\ge 0}, \quad \frac{\alpha}{m} \sum_{e \in E} \mathrm{Ent}\left[\mathbb{E}\left[F^2 \mid X_e\right]\right] \le \mathcal{E}_Q(f,f), \quad \text{with} \quad \alpha = \Omega\left(\frac{m}{\kappa \log \frac{1}{\phi}}\right),$$

where $\phi \le \min\{\mu_e(c) \mid e \in E, c \in \{0,1\}\}$ is the marginal lower bound.

Theorem

There is a family of transport flows such that $\kappa = O(\Delta^2 m)$ for P_{JS}

Construction of the transport flow Γ_e: local-flip coupling

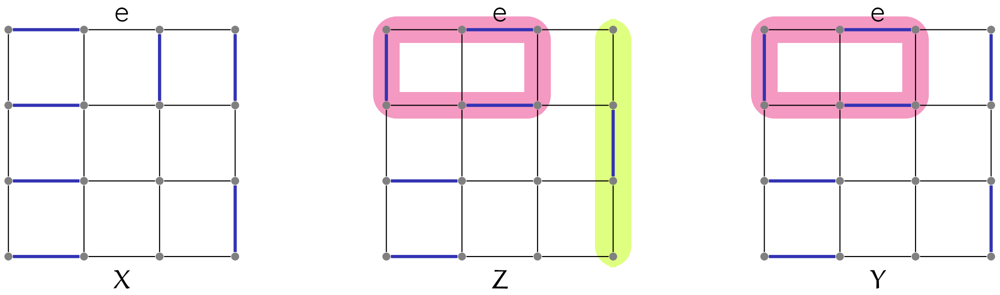
Local-flip coupling

Let X, Y be two random matchings generated as follow:

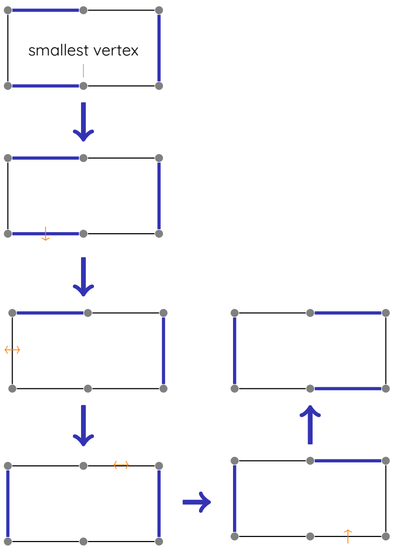
- sample $X \sim \mu^{e \leftarrow 0}$ and $Z \sim \mu^{e \leftarrow 1}$ independently
  the difference between X and Z are paths and cycles
- let D be the unique path/cycle that contains e
- let $Y = Z_D \cup X_{E \setminus D}$

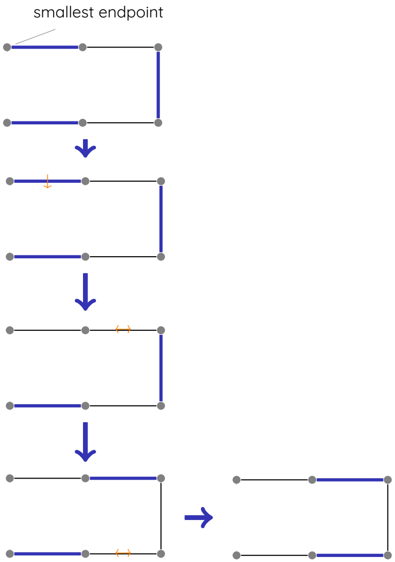
Fact

- $X \sim \mu^{e \leftarrow 0}$
- $Y \sim \mu^{e \leftarrow 1}$



Construction of the transport flow Γ_e: JS's canonical path





Construction of the transport flow Γ_e

Construction of $\{\Gamma_e \mid e \in E\}$

- let $X \sim \mu^{e\leftarrow 0}$ and $Y \sim \mu^{e\leftarrow 1}$ be sampled from the local-flip coupling
- let γ be the canonical path from X to Y, then $\mathrm{Law}(\gamma) = \Gamma_e$

Goal: $\forall$ transition $(\alpha \mapsto \beta)$ of P_{JS},

$$\sum_{e\in E} \mu_e(0)\mu_e(1) \mathop{\mathbb{E}}_{\gamma\sim\Gamma_e} [\ell(\gamma)\cdot \mathbf{1}[(\alpha\mapsto\beta)\in\gamma]] \le O(\Delta^2 m)\cdot \mu(\alpha)P_{JS}(\alpha,\beta) \qquad (\star)$$

Issue: it is hard to analysis multiple couplings together

local-flip coupling is highly symmetric

Claim

$$\mu_e(0)\mu_e(1) \Pr_{(X,Y)\sim\Gamma_e}[X=x, Y=y]$$
$$=\mu_h(0)\mu_h(1) \Pr_{(X,Y)\sim\Gamma_h}[X=x, Y=y]$$

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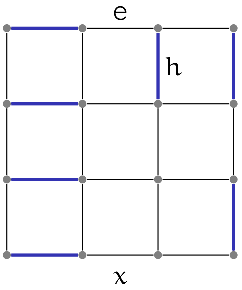

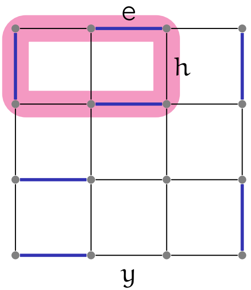

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Issue: it is hard to analysis multiple couplings together

Decoupling lemma

$\forall$ transition $(\alpha \mapsto \beta)$, if $h \in E$ s.t. $\alpha_h \neq \beta_h$, then $(\bigstar)$ holds if

$$\begin{aligned} \mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h} \left[|X \oplus Y|^2 \cdot \mathbf{1}[(\alpha \mapsto \beta) \in \gamma] \right] &\le O(\Delta^2 m) \cdot \mu(\alpha) P_{\mathrm{JS}}(\alpha, \beta) \\ &= O(\Delta^2) \cdot \min\{\mu(\alpha), \mu(\beta)\} \end{aligned}$$

Goal

$$\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^2 \cdot \mathbf{1}[(\alpha \mapsto \beta) \in \gamma]\right] \leq O(\Delta^2) \cdot \min\{\mu(\alpha), \mu(\beta)\}$$

Observation: If $\alpha_h = 0$ and $\beta_h = 1$, then $(\alpha \mapsto \beta) \in \gamma \iff Y = \beta$

$$\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^2 \cdot \mathbf{1}[Y = \beta]\right] = \mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^2 \mid Y = \beta\right] \Pr_{\Gamma_h}[Y = \beta]$$
$$= \mu_h(0)\mu(\beta) \mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^2 \mid Y = \beta\right]$$

Lemma

$$\forall h \in E, \forall p \geq 0, \quad \mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^p \mid Y = \beta\right] = O_p(\Delta^p)$$

Proved by a standard percolation analysis w.r.t. local-flip coupling

This implies

$$\Rightarrow \mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^2 \cdot \mathbf{1}[Y = \beta]\right] = \mu_h(0)\mu(\beta)O(\Delta^2) \leq O(\Delta^2)\min\{\mu(\alpha), \mu(\beta)\}$$

The $\alpha_h = 1, \beta_h = 0$ case is symmetric

Congestion analysis: $\alpha \oplus \beta = \{h\}$ $\{\downarrow, \uparrow\}$-transitions

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$$\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^2 \cdot \mathbf{1}[(\alpha \mapsto \beta) \in \gamma]\right] \le O(\Delta^2) \cdot \min\{\mu(\alpha), \mu(\beta)\}$$

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Congestion analysis: $\alpha \oplus \beta = \{h, g\}$ $\leftrightarrow$-transitions

(also works for cycles after some adjustment)

Q: how to understand $(\alpha \mapsto \beta) \in \gamma$?

Jerrum-Sinclair'89 hints a change of variable

This means we can rewrite the expectation

$$\mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^2 \cdot \mathbf{1}[(\alpha \mapsto \beta) \in \gamma]\right]$$

$$= \mathop{\mathbb{E}}_{\Gamma_h}\left[\left|\widetilde{X} \oplus \widetilde{Y}\right|^2 \cdot \mathbf{1}[\widetilde{Y} = \alpha \cup \beta]\right]$$

Q: how to understand the law of $\widetilde{X}$ and $\widetilde{Y}$?

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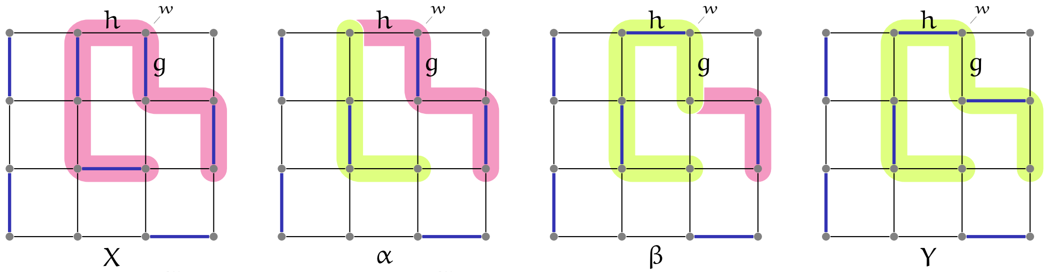

$\widetilde{X} := Y \triangle D^h$ $\widetilde{Y} := Y \triangle D^g = \alpha \cup \beta$

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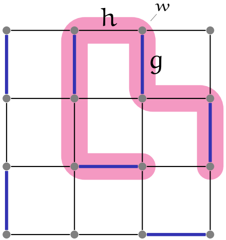
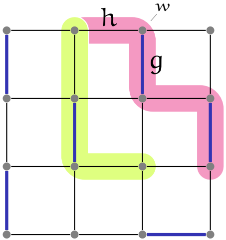
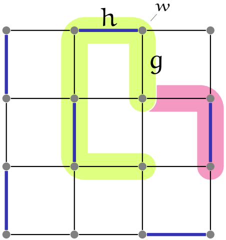


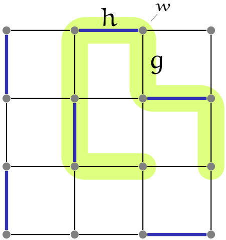
X



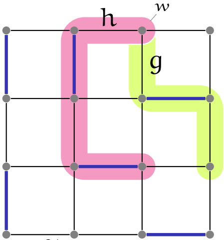
α



β



Y



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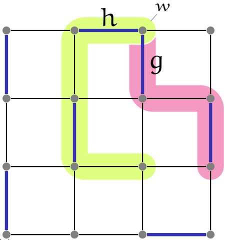

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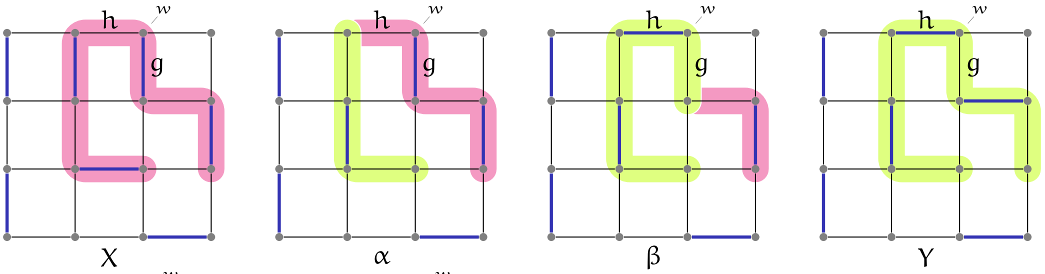

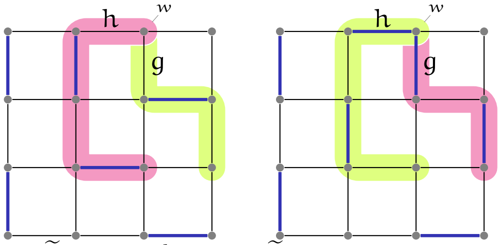

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The following is new compare to Jerrum-Sinclair's analysis

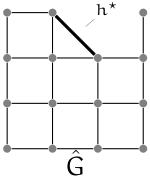
$\widetilde{X} := Y \triangle D^h$

$\widetilde{Y} := Y \triangle D^g = \alpha \cup \beta$

$\widehat{X} := \mathrm{proj}(\widetilde{X})$

$\widehat{Y} := \mathrm{proj}(\widetilde{Y})$

$(\widetilde{X}, \widetilde{Y})$ corresponds to matchings on $\hat{G}$:



delete edges around w except h, g and merge g, h into a single edge

Lemma

$$(\widetilde{X}, \widetilde{Y}) \mapsto (\widehat{X}, \widehat{Y})$$

is an injection

Lemma

transport flow $\hat{\Gamma}_{h^\star}$ for $\hat{\mu} = \mu_{\hat{G}}$ satisfies

$$\mu_h(0)\mu_h(1) \Pr_{\Gamma_h}\left[(\widetilde{X}, \widetilde{Y})\right] = O((\hat{Z}/Z)^2)\hat{\mu}_{h^\star}(0)\hat{\mu}_{h^\star}(1) \Pr_{\hat{\Gamma}_{h^\star}}\left[(\widehat{X}, \widehat{Y})\right]$$

partition functions: $Z = \frac{1}{\mu(\varnothing)}$ and $\hat{Z} = \frac{1}{\hat{\mu}(\varnothing)}$

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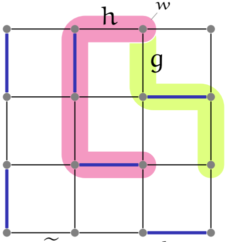

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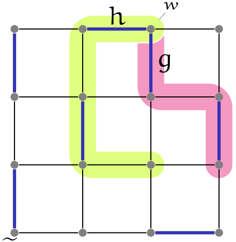

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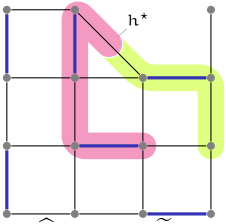

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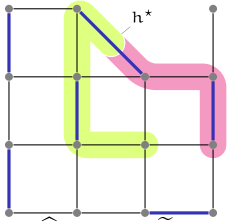

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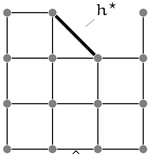

$\hat{G}$

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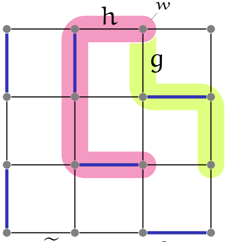

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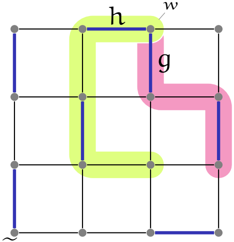
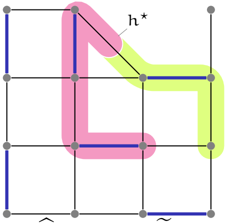
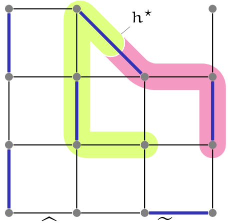


$\widetilde{Y} := Y \triangle D^g = \alpha \cup \beta$

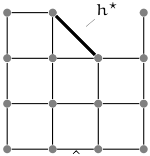


$\widehat{X} := \mathrm{proj}(\widetilde{X})$



$\widehat{Y} := \mathrm{proj}(\widetilde{Y})$

$(\widetilde{X}, \widetilde{Y})$ corresponds to matchings on $\hat{G}$:



$\hat{G}$

delete edges around w except h, g and merge g, h into a single edge

Lemma

$$(\widetilde{X}, \widetilde{Y}) \mapsto (\widehat{X}, \widehat{Y})$$

is an injection

Lemma

transport flow $\hat{\Gamma}_{h^\star}$ for $\hat{\mu} = \mu_{\hat{G}}$ satisfies

$$\mu_h(0)\mu_h(1) \Pr_{\Gamma_h}\left[(\widetilde{X}, \widetilde{Y})\right]$$

$$= O((\hat{Z}/Z)^2)\hat{\mu}_{h^\star}(0)\hat{\mu}_{h^\star}(1) \Pr_{\hat{\Gamma}_{h^\star}}\left[(\widehat{X}, \widehat{Y})\right]$$

partition functions: $Z = \frac{1}{\mu(\varnothing)}$ and $\hat{Z} = \frac{1}{\hat{\mu}(\varnothing)}$

Congestion analysis: $\alpha \oplus \beta = \{h, g\}$

(also works for cycles after some adjustment)

This means we have

$$
\begin{aligned}
&\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h} \left[|X \oplus Y|^2 \cdot \mathbf{1}[(\alpha \mapsto \beta) \in \gamma] \right] \\
=&\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h} \left[\left| \widetilde{X} \oplus \widetilde{Y} \right|^2 \cdot \mathbf{1}[\widetilde{Y} = \alpha \cup \beta] \right] \\
=&O((\hat{Z}/Z)^2) \cdot \hat{\mu}_{h^\star}(0)\hat{\mu}_{h^\star}(1) \mathop{\mathbb{E}}_{\hat{\Gamma}_{h^\star}} \left[\left| \widehat{X} \oplus \widehat{Y} \right|^2 \cdot \mathbf{1}[\widehat{Y} = \mathrm{proj}(\alpha \cup \beta)] \right]
\end{aligned}
$$

Similar to the case $\alpha \oplus \beta = \{h\}$, we have

$$\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h} \left[|X \oplus Y|^2 \cdot \mathbf{1}[(\alpha \mapsto \beta) \in \gamma] \right] = O((\hat{Z}/Z)^2) \cdot \hat{\mu}_{h^\star}(0)\hat{\mu}(\mathrm{proj}(\alpha \cup \beta))\Delta^2$$

Note that $\hat{Z} = \sum_{M \in \hat{G}} \lambda^{|M|} \leq \sum_{M \in G} \lambda^{|M|} = Z$, and

$$\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h} \left[|X \oplus Y|^2 \cdot \mathbf{1}[(\alpha \mapsto \beta) \in \gamma] \right] = O(\Delta^2) \frac{\lambda^{|\mathrm{proj}(\alpha \cup \beta)|}}{Z} = O(\Delta^2) \min\{\mu(\alpha), \mu(\beta)\}$$

Congestion analysis: $\alpha \oplus \beta = \{h, g\}$ $\leftrightarrow$-transitions

(also works for cycles after some adjustment)

This means we have

$$\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h}\left[|X \oplus Y|^2 \cdot \mathbf{1}[(\alpha \mapsto \beta) \in \gamma]\right]$$

$$=\mu_h(0)\mu_h(1) \mathop{\mathbb{E}}_{\Gamma_h}\left[\left|\widetilde{X} \oplus \widetilde{Y}\right|^2 \cdot \mathbf{1}[\widetilde{Y} = \alpha \cup \beta]\right]$$

$$=O((\hat{Z}/Z)^2) \cdot \hat{\mu}_{h^\star}(0)\hat{\mu}_{h^\star}(1) \mathop{\mathbb{E}}_{\hat{\Gamma}_{h^\star}}\left[\left|\widehat{X} \oplus \widehat{Y}\right|^2 \cdot \mathbf{1}[\widehat{Y} = \mathrm{proj}(\alpha \cup \beta)]\right]$$

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$$

Thank you

arXiv:2504.02740

Conclusion

Better bounds for Poincaré inequality and log-Sobolev inequality follows from

- low (one-sided) discrepancy coupling of $\mu^{e\leftarrow a}$ and $\mu^{e\leftarrow b}$, i.e.

  small $\mathbb{E}\left[|X \oplus Y|^p \mid Y = y\right]$ for some $p \geq 0$

- good construction of canonial paths

Future directions:

- Find more applications: e.g.
  permanent, Ising model, switch/flip chain for sampling regular graphs
  (note that it is acceptable that descrepancy $= n^c$ for some $c \in (0, 1)$)
- The relationship between local functional inequalities with SI and EI?
- Exact mixing time bound for matchings: $\widetilde{O}(\Delta^c m), c = ?$
  possible candidates: $c \in \{0, 0.5, 1, \underset{\text{this work}}{2}\}$